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SIMPLICIAL-LATTICE MODEL AND METRIC-TOPOLOGICAL CONSTRUCTIONS

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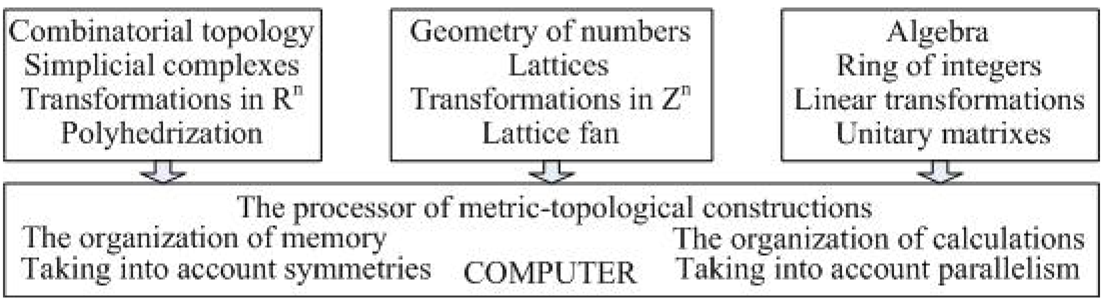
In simulation computer systems for large scientific and technical projects a computing core (set of numerical tools), environment of realization and simulation control are frequently considered from positions general metrics-topological structure and immersing such structure into real type of computer system which structure also has the specificity. Observable methods convergence in such field is caused by using many fundamental notions and achievements of combinatorial topology, differential geometry, geometry of numbers, theory of groups and graph theory. The variant of through realization of such approach from mathematical models up to the numerical results received on tool system is considered in the paper.

Introduction

Realization of discrete models for metric-topological approximations (first of all Euclidean space) is connected with two main difficulties. Firstly one is mathematical correctness of numerical interpretation for classical theoretical basis. Secondly it's necessary to get very large computing resources for receiving significant results.
The following goals were put at creation of described tool system:

1. Theoretical fundamentality of initial notions and methods [1-4].
2. Maximal usage of symmetries in the memory structure organization and calculations.
3. Getting numerical results.

The number of models for simplicial and lattice structures is created on the basis of two main sets: Z^n - set of integer points (points with integer coordinates) and Vn-set of prime edges-intervals (incidental to integer points and not having internal integer points). These models make a basis of the tool system, allowing representing, storing, analyzing, transforming and synthesizing objects as the simplicial complexes – polyhedrons, admitting the control of topological invariants, and connecting with lattice structures for approximation to the Euclidean metrics. The common diagram of positions in described questions can be presented as:



Authors leaned on the results stated in works on the following subjects: axiomatics of fininite topology [6], metric approximations on lattice and cellular structures [11, 15], models of a global polyhedrization [10], discrete analogues of homotopic transformations [9], synthesis and transformations of triangulation and mesh structures [13].

1. Transforms unit cube on itself and construction of translated polyhedrization

Splitting of 3D unit cube into six pyramids (3D simplexes) equal volume is widely used. The construction is induced by carrying out of six (collinear in pairs) diagonals in parallel faces and a "big"diagonal ((0,0,0), (1,1,1)) (Fig. 1). Such construction can be considered as result of six transforms (projecting) of a cube with a diagonal ((0,0,0), (1,1,1)) in space R^3 on subspaces (planes) R^2 in which cube faces lay.

A general view of such transforms:

$$\xi = A_i x + b_{ij},\ i=1,2,3;\ j=1,2;\ \text{(projecting along } x_i\text{)},$$

A_i - 1-diagonal square matrix with 0 on i-th position, b_{i1}-zero vector, b_{i2}-with one, distinct from 0, an element 1 on i-th position. Images of a diagonal ((0,0,0), (1,1,1)) in faces and "big" diagonal induce normal splitting of a cube on six 3D simplexes, twelve 2D simplexes (triangles in faces), twelve 1D simplexes (edges of the cube) and eight 0D simplexes (vertexes of the cube) at such projectings. The cube is a geometrical complex [1] at such splitting. The simplexes of all dimensions do not contain internal integer points, i.e. are prime simplexes.

Let's distribute the received splitting into all unit cubes in R^3 with vertexes in the integer points by parallel displacement. It's possible to speak about an infinite geometrical complex in R^3 on prime simplexes. We can associate a set (complex) of simplexes with each integer point common for the simplexes. Such complex is a simplicial star of a point and polyhedron corresponds to it. The polyhedron is strictly convex [2] with 24 faces, 14 vertexes on bound and 36 edges in this example. As pairs triangular faces lay in one plane, it's possible to consider a parallelohedron (14 tops, 24 edges and 12 faces), homeomorphic to rombododecahedron. The term cubododecahedron is used sometimes (Fig. 1, b).

Translation of anyone such polyhedron on any integer vector completely coincides with the same in our construction. Such polyhedrization R^3 is translated, and cubododecahedron itself is translated polyhedron.

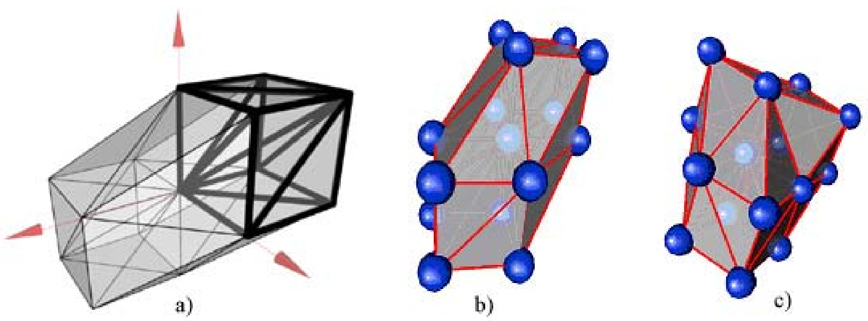


Fig. 1. Construction of simplicial star (a) and translated polyhedron (b), congruented, not convex polyhedron (c)

Similar projections in subspaces (as hyperplanes of faces) of dimensions n-1, n-2, … 2 with corresponding matrixes of transformations should be considered for n-dimensional unit cube in R^n with diagonal (1,1,…1). nD polyhedron has 2^n-2 vertexes and $2(n!+(2^{n-1}-1)(n-1)!)$ simplexes. Different types of polyhedrization are possible for R^3: congruented (coincidence by translation and rotation) (Fig. 2, c), pair-translated (polyhedrization by two types of polyhedrons) and others [14]. Homotopic transformations [5, 14] are realized on the simplicial complexes of polyhedrization.

2. Transforms Z^n on itself and construction of lattice fan

Sets of vectors corresponding to nonreducible proper fractions are used widely in models for Euclidean metric approximation in Z^2. Enumeration and ordering of such fractions are represented by Farey sequences - Φ (k) (k-order of sequence, all fractions in sequence have a denominator less k and are ordered on increase) . 0/1 and 1/1 are correct nonreducible fractions under definition [3]. The basic graph construction for each point in Z^2 consists of the vectors (as edges) corresponding Φ (k) for $\{0,\pi/4\}$ and symmetric images around the point. It's possible to approach the shortest paths between integer points (as vertexes) to Euclidean length on such graphs with growing k [7, 12].

It is possible to consider the construction of such fan of prime vectors around (0,0) as set of transforms Z^2 on the own subsets. We shall designate through $Z^2\{\varphi_1,\varphi_2\}$ at $\varphi_2 > \varphi_1$; φ_2, $\varphi_1 < \pi/2$ set of the integer points in sector $\varphi_2 - \varphi_1$. Then the vectors, corresponding to fractions Φ (k) (numerator - coordinate y, denominator - coordinate x) will break set $Z^2\{0,\pi/4\}$ on N (k)-1 sectors, where N (k) - number of members in Φ(k). Let be two neighboring members in Φ(k) are equal a_i/b_i and a_{i+1}/b_{i+1} and $\varphi_i= \arctan(a_i/b_i)$, $\varphi_{i+1}= \arctan(a_{i+1}/b_{i+1})$,
$i = 1,2,..,N(k)-1$. We shall consider transformation $Z^2\{0,\pi/2\}$ (with all 1-edges between the integer points), represented by a matrix:

$$A_i = \begin{pmatrix} b_i & b_{i+1} \\ a_i & a_{i+1} \end{pmatrix}.$$

It is not difficult to be convinced, that $Z^2\{\varphi_i,\varphi_{i+1}\}= A_i\, Z^2\{0,\pi/2\}$ and from the basic property of next fractions Φ(k) follows $|a_{i+1}b_i - a_ib_{i+1}|=1$, that corresponds $|\,|A_i|\,|=1$. As elements of matrix A_i are integers and determinant equals 1, it's unitary matrix above a ring of integers with all inherent group properties [4]. We shall note only the most essential property: such transformation keeps the areas, i.e. each unit square from Z^2 transforms in a parallelogram with vertexes - the integer points and the area equals 1. Thus, the set $\{A_i\, Z^2 \{0,\pi/2\}\}$ is set of lattices in sectors with the basic vectors corresponding to the neighboring fractions in Φ(k), and covers $Z^2\{0,\pi/4\}$: $Z^2\{0,\pi/4\} = \bigcup_{i=1}^{N(k)-1} A_i\, Z^2\{0,\pi/2\}$.

Symmetric display (also linear transformation) concerning a line y=x delivers construction in $Z^2\{\pi/4,\pi/2\}$. Then three rotations of set $\{ Z^2\{0,\pi/4\} \cup Z^2\{\pi/4,\pi/2\}\}$ complete construction of lattice fan around (0,0) (Fig. 2, a). Construction is translated in each point Z^2. It is possible to consider the formed graph as nonoriented. The shortest path between the integer points is determined as graph edges with weight (Euclidean length). Analogue of nondeductible fraction in a multidimensional case will be vector with components - integer numbers, not having common divider more 1. It is possible to set the more general problem. Let Δ is maximal relative inaccuracy between the shortest path in lattice fan and Euclidean length for any pair of the integer points. It's necessary to construct set of matrixes of analogue transformations and to determine all basic vectors for such lattice fan by that. Such constructions are realized by a method fan triangulation, analogue of nonreducible fractions generation [8]. 3D basic vectors and faces between them, forming a triangulation under Δ=0.01, are shown by projections to unit sphere (Fig. 2, b), and for 4D and Δ=0.1 projection - first on 4D unit sphere, and then in 3D subspace, the triangulation is located inside a unit 3D ball (Fig. 2, c).

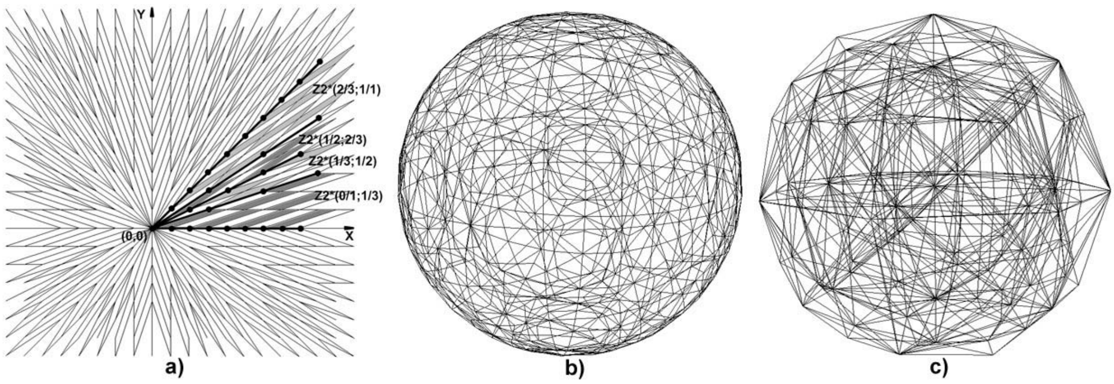


Fig. 2. 2D lattice fan for Φ(3) (a), sphere projecting of 3D lattice fan Δ=0.01 (b), double projecting (4D unit sphere, 3D unit ball) of 4D lattice fan Δ=0.1 (c)

3. Tool system

The system is written in C++ for windows platform. The following basic macro operations are realized in system as the prototype of the topological processor:

1. System adjustment depending on sizes of general lattice and polyhedron type.
2. Generation lattice fan by given inaccuracy Δ.
3. Displays set of barrier - defects in simplicial-lattice model.
4. Running of a metric wave on set of lattice fans (vertexes weighting for search of the shortest paths from a set – source) and construction of equidistant graph (set of achievable vertexes with weights and edges on which the shortest paths are realized).
5. Compression and expansion of simplicial complexes without destruction of topological invariants under given conditions.
6. Allocation triangular bounds in simplicial complex.
7. Displaying by OpenGL and VRML software tools.

The basic features for a desktop class:

1. The allowable sizes of lattices are up to 200x200x200. 2. Translated constructions give possibility to store in memory the only copy of simplicial star (polyhedron) and 1/48 ($1/2^3 3!$) part of the only lattice fan copy. 3. Due to tabulated storage in operative memory of all possible (2^{14}=16K) situations on bound of polyhedron, operation of the topological analysis is carried out for one step. 4. Visualization by means of OpenGL and VRML.

As a mini-example of a variational problem, we shall consider the following statement. To construct triangular sphere as a removed on equal distance (in view of detour of the set barrier) from the center (x, y, z) set under given Δ-relative inaccuracy of difference from Euclidean metrics and provided that sphere has no of barrier elements. At the given statement consistently all macro operations of the topological processor 1-2-3-4-5-6-7 are carried out. For a lattice 50x50x50, the center (21,18,21) and two barrier - parallelepipeds for the decision of a problem on PC (Intel Celeron 2,66 GHz, 512 MB RAM) was required on macro operations: 1-3 <0,01 s; 4-33s; 5-61s; 6-29s; 7-1s. One of possible decisions is shown on (Fig. 3).

Tool system was applied effectively in 3d tomography segmentation and generation of minimal surfaces.

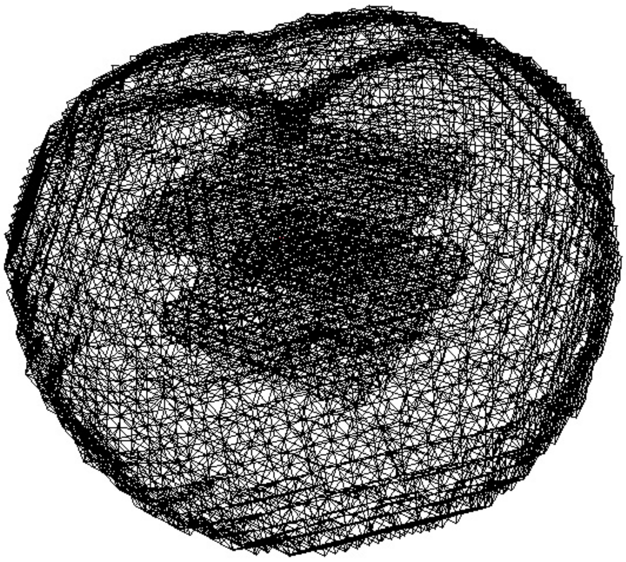


Fig. 3. Triangular sphere ("apple") is macro operations result of topological processor in lattice 50x50x50. Centre is located between two right-angle barriers. Δ=0.01

Conclusion

Completed emulation of operations on desktop and universality of main models give possibility ND realizations (N=4,5,6) in super computer.

System transfer on the super computer cluster of the Moscow State University is planned with the purpose:

1. Opportunities of the 3D problems decision on lattices up to 2000x2000x2000 (2000^3), 4D-300^4, 5D-100^5, 6D-50^6.
2. Realizations of the parallelism potentially close to cellular automatic devices.
3. Connection of the 4D, 5D, 6D visualization block.

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